Block Toeplitz operators with rational symbols (II)

2008 J. Phys. A: Math. Theor. 41 385206

(http://iopscience.iop.org/1751-8121/41/38/385206)

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **41** (2008) 385206 (7pp) doi:10.1088/1751-8113/41/38/385206

Block Toeplitz operators with rational symbols (II)

In Sung Hwang[1] **and Woo Young Lee**[2]

[1] Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Korea
[2] Department of Mathematics, Seoul National University, Seoul 151-742, Korea

E-mail: ishwang@skku.edu and wylee@math.snu.ac.kr

Received 15 May 2008, in final form 29 July 2008
Published 28 August 2008
Online at stacks.iop.org/JPhysA/41/385206

Abstract
In this paper we derive a formula for the rank of the self-commutator of hyponormal block Toeplitz operators T_Φ with matrix-valued rational symbols Φ in $L^\infty(\mathbb{C}^{n\times n})$ via the classical Hermite-Fejér interpolation problem.

PACS numbers: 02.30.Sa, 02.30.Tb
Mathematics Subject Classification: 47B35, 47B20, 47A57, 46B70

1. Introduction

Block Toeplitz operators are of importance in connection with a variety of problems in the field of quantum mechanics. Also, a study of spectral properties of hyponormal operators has made important contributions in the study of related mathematical physics problem. The hyponormality of block Toeplitz operators with rational symbols was considered in [HL]. This paper is a continuation of [HL]: the rank formula for the self-commutators of hyponormal block Toeplitz operators is derived here. The rank of the self-commutator plays an important role in the model theory of hyponormal operators.

A bounded linear operator A on an infinite-dimensional complex Hilbert space $\mathcal{H}$ is said to be *hyponormal* if its self-commutator $[A^*, A] = A^*A - AA^*$ is positive (semidefinite). For φ in $L^\infty(\mathbb{T})$ of the unit circle $\mathbb{T} = \partial\mathbb{D}$, the (single) Toeplitz operator with symbol φ is the operator T_φ on the Hardy space $H^2(\mathbb{T})$ defined by

$$T_\varphi f = P(\varphi f) \qquad (f \in H^2(\mathbb{T})),$$

where P denotes the orthogonal projection that maps from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

For the matrix-valued function $\Phi \in L^\infty(\mathbb{C}^{n\times n})$, the *block Toeplitz operator with symbol* Φ is the operator T_Φ on the vector-valued Hardy space $H^2(\mathbb{C}^n)$ of the unit disc defined by

$$T_\Phi h = P_n(\Phi h) \qquad (h \in H^2(\mathbb{C}^n)),$$

where P_n denotes the orthogonal projection that maps $L^2(\mathbb{C}^n)$ onto $H^2(\mathbb{C}^n)$. If we set $H^2(\mathbb{C}^n) = H^2(\mathbb{T}) \oplus \cdots \oplus H^2(\mathbb{T})$ then we see that if

$$\Phi = \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ & \vdots & \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{bmatrix}$$

then

$$T_\Phi = \begin{bmatrix} T_{\varphi_{11}} & \cdots & T_{\varphi_{1n}} \\ & \vdots & \\ T_{\varphi_{n1}} & \cdots & T_{\varphi_{nn}} \end{bmatrix}.$$

The block Hankel operator with symbol $\Phi \in L^\infty(\mathbb{C}^{n\times n})$ is the operator $H_\Phi : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^n)$ defined by

$$H_\Phi h = J_n(I - P_n)(\Phi h),$$

where J_n denotes the unitary operator from $H^2(\mathbb{C}^n)^\perp$ to $H^2(\mathbb{C}^n)$ given by $J_n(z^{-m}) = z^{m-1}$ for $m \geqslant 1$. For $\Phi \in L^\infty(\mathbb{C}^{n\times n})$ write $\widetilde{\Phi}(z) := \Phi^*(\overline{z})$. An inner matrix $\Theta(z) \in H^2(\mathbb{C}^{n\times m})$ is the one satisfying $\Theta(z)^*\Theta(z) = I_m$ for all $z \in \mathbb{T}$, where I_m the $m \times m$ identity matrix. The following relations can easily be proved:

$$T_\Phi^* = T_{\Phi^*}, H_\Phi^* = H_{\widetilde{\Phi}} \qquad (\Phi \in L^\infty(\mathbb{C}^{n\times n})), \tag{1.1}$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H_{\Phi^*}^* H_\Psi \qquad (\Phi, \Psi \in L^\infty(\mathbb{C}^{n\times n})), \tag{1.2}$$

$$H_\Phi T_\Psi = H_{\Phi\Psi}, H_{\Psi\Phi} = T_{\widetilde{\Psi}}^* H_\Phi \qquad (\Phi \in L^\infty(\mathbb{C}^{n\times n}), \Psi \in H^\infty(\mathbb{C}^{n\times n})), \tag{1.3}$$

$$H_\Phi^* H_\Phi - H_{\Theta\Phi}^* H_{\Theta\Phi} = H_\Phi^* H_{\Theta^*} H_{\Theta^*}^* H_\Phi \qquad (\Phi \in L^\infty(\mathbb{C}^{n\times n}), \Theta \in H^\infty(\mathbb{C}^{n\times n})\text{inner}). \tag{1.4}$$

The problem of determining which symbols induce hyponormal block Toeplitz operators was solved in [GHR] by the aid of Cowen's theorem [Co].

Theorem 1.1 [GHR]**.** *For $\Phi \in L^\infty(\mathbb{C}^{n\times n})$, T_Φ is hyponormal if and only if Φ is normal and*

$$\mathcal{E}(\Phi) := \{K \in H^\infty(\mathbb{C}^{n\times n}) : ||K||_\infty \leqslant 1 \text{ and } \Phi - K\Phi^* \in H^\infty(\mathbb{C}^{n\times n})\}$$

is nonempty.

For $\Phi \in L^\infty(\mathbb{C}^{n\times n})$ write

$$\Phi_+ := P(\Phi) \in H^2(\mathbb{C}^{n\times n}) \qquad \text{and} \qquad \Phi_- := [(I-P)(\Phi)]^* \in H^2(\mathbb{C}^{n\times n}),$$

where P denotes the orthogonal projection from $L^2(\mathbb{C}^{n\times n})$ to $H^2(\mathbb{C}^{n\times n})$. Thus we can write $\Phi = \Phi_-^* + \Phi_+$. For an inner matrix Θ, write $\mathcal{H}(\Theta) = (\Theta(z)H^2(\mathbb{C}^n))^\perp$. For $F = [f_{ij}] \in H^\infty(\mathbb{C}^{n\times n})$, we say that F is called *rational* if each entry f_{ij} is a rational function. Also if given $\Phi \in L^\infty(\mathbb{C}^{n\times n})$, Φ_+ and Φ_- are rational then we say that the block Toeplitz operator T_Φ has a rational symbol Φ.

The case of arbitrary matrix symbol $\Phi \in L^\infty(\mathbb{C}^{n\times n})$, though solved by theorem 1.1, is in practice very difficult because the matrix multiplication is not commutative. In [HL], it was shown that if $\Phi \in L^\infty(\mathbb{C}^{n\times n})$ is a rational symbol then the hyponormality of the block Toeplitz operator T_Φ can be determined by the matrix-valued tangential Hermite-Fejér interpolation problem. However this criterion does not give any information on the rank of the self-commutator. In this paper we derive a formula on the rank of the self-commutator of hyponormal block Toeplitz operators T_Φ with matrix-valued rational symbols Φ in $L^\infty(\mathbb{C}^{n\times n})$ via the classical Hermite-Fejér interpolation problem.

2. The main result

In view of [HL, lemma 2.1], when we study hyponormal block Toeplitz operators with rational symbols Φ we may assume that the symbol $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty(\mathbb{C}^{n\times n})$ is of the form

$$\Phi_+ = [\theta_1\theta_0\overline{a_{ij}}] = \Theta_1(z)\Theta_0(z)A^*(z) \qquad \text{and} \qquad \Phi_- = [\theta_1\overline{b_{ij}}] = \Theta_1(z)B^*(z),$$

where $\Theta_i = \theta_i I_n$ $(i = 0, 1)$ and the θ_i are finite Blaschke products and $A, B \in H^\infty(\mathbb{C}^{n\times n})$.

Let θ be a finite Blaschke product of degree d:

$$\theta = \mathrm{e}^{\mathrm{i}\xi}\prod_{i=1}^{N}(\widetilde{B}_i)^{m_i} \qquad \left(\widetilde{B}_i := \frac{z-\alpha_i}{1-\overline{\alpha_i}z}\right), \tag{2.1}$$

where $d = \sum_{i=1}^N m_i$. For our purpose rewrite θ as in the form

$$\theta = \mathrm{e}^{\mathrm{i}\xi}\prod_{j=1}^{d} B_j, \tag{2.2}$$

where

$$B_j := \widetilde{B_k} \qquad \text{if} \qquad \sum_{l=0}^{k-1} m_l < j \leqslant \sum_{l=0}^{k} m_l$$

and, for notational convenience, $m_0 := 0$. Let

$$\phi_j := \frac{q_j}{1-\overline{\alpha_j}z}B_{j-1}B_{j-2}\cdots B_1 \qquad (1 \leqslant j \leqslant d), \tag{2.3}$$

where $\phi_1 := q_1(1-\overline{\alpha_1}z)^{-1}$ and $q_j := (1-|\alpha_j|^2)^{\frac{1}{2}}$ $(1 \leqslant j \leqslant d)$. It is well known that $\{\phi_j\}_{j=1}^d$ is an orthonormal basis for $\mathcal{H}(\theta)$.

For our purpose we concentrate on the data given by sequences of $n\times n$ complex matrices. Given the sequence $\{K_{ij} : 1 \leqslant i \leqslant N, 0 \leqslant j < m_i\}$ of $n \times n$ complex matrices and a set of distinct complex numbers $\alpha_1, \dots, \alpha_N$ in $\mathbb{D}$, the classical Hermite-Fejér interpolation problem is to find necessary and sufficient conditions for the existence of a contractive analytic function K in $H^\infty(\mathbb{C}^{n\times n})$ satisfying

$$\frac{K^{(j)}(\alpha_i)}{j!} = K_{i,j} \qquad (1 \leqslant i \leqslant N, 0 \leqslant j < m_i). \tag{2.4}$$

To construct a polynomial $K(z) \equiv P(z)$ satisfying (2.4), let $p_i(z)$ be the polynomial of order $d - m_i$ defined by

$$p_i(z) := \prod_{k=1,k\neq i}^{N}\left(\frac{z-\alpha_k}{\alpha_i-\alpha_k}\right)^{m_k}.$$

Consider the polynomial $P(z)$ of degree $d-1$ defined by

$$P(z) := \sum_{i=1}^{N}\left(K'_{i,0} + K'_{i,1}(z-\alpha_i) + K'_{i,2}(z-\alpha_i)^2 + \cdots + K'_{i,m_i-1}(z-\alpha_i)^{m_i-1}\right)p_i(z), \tag{2.5}$$

where the $K'_{i,j}$ are obtained by the following equations:

$$K'_{i,j} = K_{i,j} - \sum_{k=0}^{j-1}\frac{K'_{i,k}p_i^{(j-k)}(\alpha_i)}{(j-k)!} (1 \leqslant i \leqslant N; 0 \leqslant j < m_i)$$

and $K'_{i,0} = K_{i,0}$ $(1 \leqslant i \leqslant N)$. Then $P(z)$ satisfies (2.4).

On the other hand, if F is a matrix-valued function in $H^\infty(\mathbb{C}^{n\times n})$, let $\mathfrak{A}(F)$ be the operator on $\mathcal{H}(\theta I_n)$ defined dy

$$\mathfrak{A}(F) := P_{\mathcal{H}(\theta I_n)} M_F|_{\mathcal{H}(\theta I_n)},$$

where M_F is the multiplication operator with symbol F. Now let W be the unitary operator from $\bigoplus_1^d \mathbb{C}^n$ onto $\mathcal{H}(\theta I_n)$ defined by

$$W := (\phi_1 I_n, \phi_2 I_n, \ldots, \phi_d I_n),$$

where ϕ_j are the functions in (2.3). Let M be the matrix on $\mathbb{C}^d$ corresponding to the finite Blaschke product θ of order d written in the form (2.2):

$$M := \begin{bmatrix} \alpha_1 & 0 & 0 & 0 & \cdots & 0 \\ q_1q_2 & \alpha_2 & 0 & 0 & \cdots & 0 \\ -q_1\overline{\alpha_1}q_3 & q_2q_3 & \alpha_3 & 0 & \cdots & 0 \\ q_1\overline{\alpha_2\alpha_3}q_4 & -q_2\overline{\alpha_3}q_4 & q_3q_4 & \alpha_4 & \cdots & 0 \\ -q_1\overline{\alpha_2\alpha_3\alpha_4}q_5 & q_2\overline{\alpha_3\alpha_4}q_5 & -q_3\overline{\alpha_4}q_5 & q_4q_5 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ (-1)^d q_1\left(\prod_{j=2}^{d-1}\overline{\alpha_j}\right)q_d & (-1)^{d-1}q_2\left(\prod_{j=3}^{d-1}\overline{\alpha_j}\right)q_d & \cdots & \cdots & q_{d-1}q_d & q_d \end{bmatrix}. \tag{2.6}$$

If L is a matrix on $\mathbb{C}^n$ and $M = [m_{i,j}]_{d\times d}$, then the matrix $L \otimes M$ is the matrix on $\mathbb{C}^{n\times d}$ defined by the block matrix

$$L \otimes M := \begin{bmatrix} Lm_{1,1} & Lm_{1,2} & \cdots & Lm_{1,d} \\ Lm_{2,1} & Lm_{2,2} & \cdots & Lm_{2,d} \\ \vdots & \vdots & \vdots & \vdots \\ Lm_{d,1} & Lm_{d,2} & \cdots & Lm_{d,d} \end{bmatrix}.$$

Now let $P(z) \in L^\infty(\mathbb{C}^{n\times n})$ be the polynomial defined by equation (2.5). The matrix $P(M)$ on $\mathbb{C}^{n\times d}$ is defined by

$$P(M) := \sum_{i=0}^{d-1} P_i \otimes M^i \qquad \text{where} \quad P(z) = \sum_{i=0}^{d-1} P_i z^i.$$

Then $P(M)$ is called the *Hermite-Fejér matrix* determined by (2.4). It is well known ([FF], theorem 5.6) that

$$W^*\mathfrak{A}(P)W = P(M), \tag{2.7}$$

which says that $P(M)$ is a matrix representation for $\mathfrak{A}(P)$.

Our main result now follows.

Theorem 2.1. *Let* $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty(\mathbb{C}^{n\times n})$ *be a normal rational symbol, with*

$$\Phi_+ = [\theta_1\theta_0\overline{a_{ij}}] = \Theta_1(z)\Theta_0(z)A^*(z) \qquad \text{and} \qquad \Phi_- = [\theta_1\overline{b_{ij}}] = \Theta_1(z)B^*(z),$$

where $\Theta_i = \theta_i I_n$ $(i = 0, 1)$ *and* θ_i *are finite Blaschke products. If* T_Φ *is hyponormal then the rank of the self-commutator* $[T_\Phi^*, T_\Phi]$ *is computed from the formula*

$$\text{rank}[T_\Phi^*, T_\Phi] = rank\left(\mathfrak{A}(A)^* W\left(I_{\mathcal{H}(\Theta_1\Theta_0)} - P(M)^*P(M)\right)W^*\mathfrak{A}(A)\right). \tag{2.8}$$

Hence, in particular, if $A(\alpha)$ *is invertible for each zero* α *of* $\theta_1\theta_0$ *then*

$$\text{rank}[T_\Phi^*, T_\Phi] = \text{rank}\left(I_{\mathcal{H}(\Theta_1\Theta_0)} - P(M)^*P(M)\right). \tag{2.9}$$

Proof. Suppose that T_Φ is hyponormal. Without loss of generality we may assume that

$$\theta_1\theta_0 = \prod_{i=1}^{N}\left(\frac{z-\alpha_i}{1-\overline{\alpha_i}z}\right)^{p_i} \qquad \text{and} \qquad \theta_1 = \prod_{i=1}^{N_1}\left(\frac{z-\alpha_i}{1-\overline{\alpha_i}z}\right)^{p_i},$$

where $d_1 = \sum_{i=1}^{N_1} p_i$ and $d_2 = \sum_{i=N_1+1}^{N} p_i$. By theorem 1.1, there exists a matrix-valued function $K(z)$ in $H^\infty(\mathbb{C}^{n\times n})$ such that

$$\Phi - K\Phi^* \in H^\infty(\mathbb{C}^{n\times n}),$$

or equivalently,

$$\Theta_0(z)B(z) - K(z)A(z) \in \Theta(z)H^2(\mathbb{C}^{n\times n}) \qquad (\Theta := \Theta_1\Theta_0). \tag{2.10}$$

Note that (2.10) holds if and only if the following equations hold: for each $i = 1, \ldots, N$,

$$\begin{bmatrix} B_{i,0}\\ B_{i,1}\\ B_{i,2}\\ \vdots\\ B_{i,m_i-2}\\ B_{i,m_i-1}\end{bmatrix} = \begin{bmatrix} K_{i,0} & 0 & 0 & 0 & \cdots & 0\\ K_{i,1} & K_{i,0} & 0 & 0 & \cdots & 0\\ K_{i,2} & K_{i,1} & K_{i,0} & 0 & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\ K_{i,m_i-2} & K_{i,m_i-3} & \ddots & \ddots & K_{i,0} & 0\\ K_{i,m_i-1} & K_{i,m_i-2} & \ldots & K_{i,2} & K_{i,1} & K_{i,0}\end{bmatrix}\begin{bmatrix} A_{i,0}\\ A_{i,1}\\ A_{i,2}\\ \vdots\\ A_{i,m_i-2}\\ A_{i,m_i-1}\end{bmatrix}, \tag{2.11}$$

where

$$K_{i,j} := \frac{K^{(j)}(\alpha_i)}{j!}, \qquad A_{i,j} := \frac{A^{(j)}(\alpha_i)}{j!} \qquad \text{and} \qquad B_{i,j} := \frac{(\theta_0 B)^{(j)}(\alpha_i)}{j!}.$$

Thus K is a function in $H^\infty(\mathbb{C}^{n\times n})$ for which

$$\frac{K^{(j)}(\alpha_i)}{j!} = K_{i,j} \qquad (1 \leqslant i \leqslant N, 0 \leqslant j < m_i), \tag{2.12}$$

where $K_{i,j}$ are determined by equation (2.11).

On the other hand, since $\Phi^*\Phi - \Phi\Phi^* = 0$, we have

$$[T_\Phi^*, T_\Phi] = H_{\Phi_+^*}^* H_{\Phi_+^*} - H_{\Phi_-^*}^* H_{\Phi_-^*} + T_{\Phi^*\Phi-\Phi\Phi^*} = H_{A\Theta_0^*\Theta_1^*}^* H_{A\Theta_0^*\Theta_1^*} - H_{B\Theta_1^*}^* H_{B\Theta_1^*}.$$

Observe that

$$\text{cl ran}(H_{A\Theta_0^*\Theta_1^*}^* H_{A\Theta_0^*\Theta_1^*}) = \text{cl ran} H_{A\Theta_0^*\Theta_1^*}^* \subseteq (\Theta_1(z)\Theta_0(z)H^2(\mathbb{C}^n))^\perp = \mathcal{H}(\Theta_1\Theta_0) = \mathcal{H}(\Theta)$$

and

$$\text{cl ran}(H_{B\Theta_1^*}^* H_{B\Theta_1^*}) \subseteq (\Theta_1(z)H^2(\mathbb{C}^n))^\perp = \mathcal{H}(\Theta_1),$$

which implies that $\text{ran}[T_\Phi^*, T_\Phi] \subseteq \mathcal{H}(\Theta)$. Thus we can see that $\mathcal{H}(\Theta)$ is a reducing subspace of $[T_\Phi^*, T_\Phi]$. Let U and V be in $\mathcal{H}(\Theta)$. Suppose $P \equiv K$ is a polynomial satisfying (2.12). Since $\ker H_{\Theta^*} = \Theta H^2(\mathbb{C}^n)$, we know that $H_{\Theta^* P}U = H_{\Theta^*}(P_{\mathcal{H}(\Theta)}(PU))$. Since $H_{\Theta^*}^* H_{\Theta^*}$ is the projection onto $\mathcal{H}(\Theta)$, it follows that

$$\langle H_{\Theta^* P}^* H_{\Theta^* P}U, V\rangle = \langle H_{\Theta^* P}U, H_{\Theta^* P}V\rangle = \langle P_{\mathcal{H}(\Theta)}PU, P_{\mathcal{H}(\Theta)}PV\rangle = \langle \mathfrak{A}(P)U, \mathfrak{A}(P)V\rangle.$$

Thus by (2.7) we have that

$$H_{\Theta^* P}^* H_{\Theta^* P}|_{\mathcal{H}(\Theta)} = \mathfrak{A}(P)^*\mathfrak{A}(P) = WP(M)^*P(M)W^*,$$

which implies

$$(H_{\Theta^*}^* H_{\Theta^*} - H_{\Theta^* P}^* H_{\Theta^* P})|_{\mathcal{H}(\theta)} = W(I_{\mathcal{H}(\Theta)} - P(M)^*P(M))W^*.$$

Since P satisfies equality (2.11) and hence $\Phi_-^* - P\Phi_+^* \in H^\infty(\mathbb{C}^{n\times n})$, it follows that

$$[T_\Phi^*, T_\Phi]|_{\mathcal{H}(\Theta)} = \left(H_{\Phi_+^*}^* H_{\Phi_+^*} - H_{\Phi_-^*}^* H_{\Phi_-^*}\right)\big|_{\mathcal{H}(\Theta)} = T_A^*\left(H_{\Theta^*}^* H_{\Theta^*} - H_{\Theta^* P}^* H_{\Theta^* P}\right)T_A\big|_{\mathcal{H}(\Theta)}$$
$$= \mathfrak{A}(A)^* W(I_{\mathcal{H}(\Theta)} - P(M)^* P(M))W^*\mathfrak{A}(A),$$

which proves (2.8). On the other hand, suppose $A(\alpha)$ is invertible for all zeros α of $\theta_1\theta_0$. If $\mathfrak{A}(A)F = 0$ for some $F \in \mathcal{H}(\Theta)$ then $P_{\mathcal{H}(\Theta)}(AF) = 0$ and hence $AF \in \Theta H^2$. Therefore $F(\alpha) = 0$ for all zeros α of Θ. It thus follows that $F \in \Theta H^2$ and hence $F \in \Theta H^2 \cap \mathcal{H}(\Theta) = \{0\}$. Thus $\mathfrak{A}(A)$ is injective. Since $\mathfrak{A}(A)$ is a finite-dimensional operator (because θ is a finite Blaschke product), we have that $\mathfrak{A}(A)$ is invertible. Therefore we get (2.9). □

We conclude with a revealing example.

Example 2.2. Let $b(z) := \frac{z-\frac{1}{2}}{1-\frac{1}{2}z}$ and let

$$T_\Phi \equiv \begin{bmatrix} T_b^* & T_{zb}^* + T_{2zb} \\ T_{zb}^* + T_{2zb} & T_b^* \end{bmatrix}.$$

Observe that

$$\Phi(z) := \begin{bmatrix} \overline{b(z)} & \overline{zb(z)} + 2zb(z) \\ \overline{zb(z)} + 2zb(z) & \overline{b(z)} \end{bmatrix} \in L^\infty(\mathbb{C}^{2\times 2}).$$

Using the notation in the preceding argument we write

$$\Phi_+(z) = zb(z)\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}^* \qquad \text{and} \qquad \Phi_-(z) = zb(z)\begin{bmatrix} z & 1 \\ 1 & z \end{bmatrix}^*.$$

Thus we can write

$$\Theta(z) = zb(z)I_2, \qquad A(z) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \qquad B(z) = \begin{bmatrix} z & 1 \\ 1 & z \end{bmatrix},$$

so that

$$\phi_1(z) = 1, \qquad \phi_2(z) = \frac{\sqrt{3}}{2}\times\frac{z}{1-\frac{1}{2}z} \qquad \text{and} \qquad M = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ \sqrt{3} & 1 \end{bmatrix}.$$

Since

$$K_{1,0} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad K_{2,0} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

it follows that

$$P(z) = K_{1,0}{}'p_1(z) + K_{2,0}{}'p_2(z)$$
$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}(-2z+1) + \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}(2z)$$
$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}z.$$

Therefore the Hermite-Fejér matrix $P(M)$ is given by

$$P(M) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bigotimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bigotimes \frac{1}{2}\begin{bmatrix} 0 & 0 \\ \sqrt{3} & 1 \end{bmatrix}$$
$$= \frac{1}{4}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & \sqrt{3} & 2 & 1 \\ \sqrt{3} & 0 & 1 & 2 \end{bmatrix}.$$

Hence a straightforward calculation shows that

$$I - P(M)^* P(M) = \frac{1}{16}\begin{bmatrix} 9 & 0 & -\sqrt{3} & -2\sqrt{3} \\ 0 & 9 & -2\sqrt{3} & -\sqrt{3} \\ -\sqrt{3} & -2\sqrt{3} & 11 & -4 \\ -2\sqrt{3} & -\sqrt{3} & -4 & 11 \end{bmatrix}$$

and hence

$$\text{rank}[T_\Phi^*, T_\Phi] = \text{rank}(I - P(M)^* P(M)) = 4.$$

Acknowledgments

The first-named author was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2007-313-C00039). The second-named author was supported by a grant (KRF-0409-20080061) from the Korea Research Foundation.

References

[Co] Cowen C 1988 Hyponormality of Toeplitz operators *Proc. Am. Math. Soc.* **103** 809–12

[FF] Foias C and Frazo A 1993 The commutant lifting approach to interpolation problems *Operator Theory: Adv. Appl.* vol 44 (Boston, MA: Birkhäuser)

[GHR] Gu C, Hendricks J and Rutherford D 2006 Hyponormality of block Toeplitz operators *Pac. J. Math.* **223** 95–111

[HL] Hwang I S and Lee W Y 2008 Block Toeplitz operators with rational symbols *J. Phys. A: Math. Theor.* **41** 521–7